Solution of the Diffusion Equation by the Finite Difference Method

This document contains a brief guide to using an Excel spreadsheet for solving the diffusion equation[1] by the finite difference method[2]. The equation that we will be focusing on is the one-dimensional simple diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2},$$

with $0 \le t \le T$ and $u(x,0) = f(x)$ is the initial condition and the goal is to find the solution for $0 < t \le T$ and in particular the final solution $u(x,T)$. We also require information about the solution at the edges of the region; $x=0$ and $x=$The coefficient $D(\ge 0)$ represents the rate of diffusion. Note that D need not be a constant, it may for example be a function of position or time, or it may be a function of u for example.

The most straightforward FDM to apply to the problem is to be obtained through replacing the derivative with respect to t by a forward difference

$$\frac{\partial u(x,t)}{\partial t} \approx \frac{u(x,t+k) - u(x,t)}{k},$$

where k is the length of the time step, and replacing the second derivative with respect to x by a central difference formula

$$\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{u(x+h,t) - 2u(h,t) + u(x-h,t)}{r^2}$$

where h represents the spatial step.

By substituting the two approximations into the original diffusion equation, we obtain the following approximate relationship between neighbouring solution values in the domain:

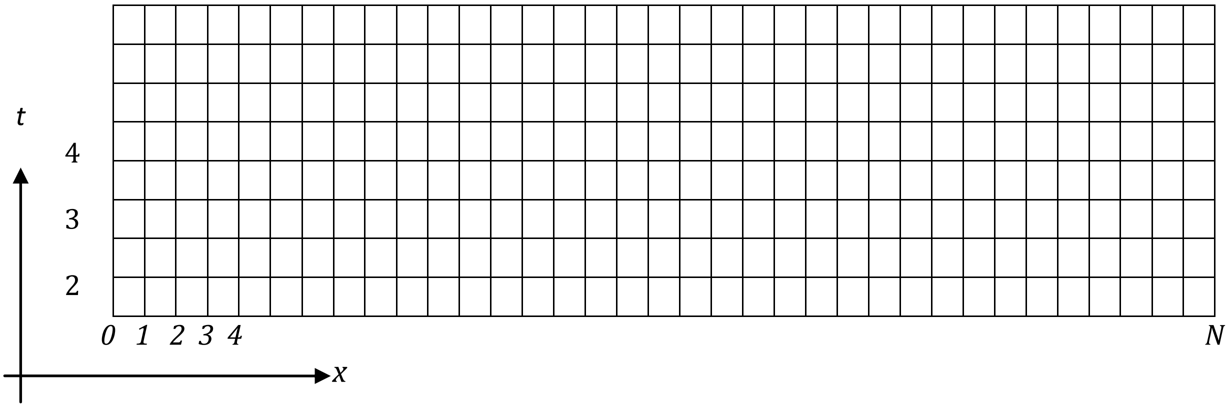
$$\frac{u(x,t+k) - u(x,t)}{k} \approx D\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

In the approximation above, there is one term at the upper time level $t+k$. Hence given the initial values at the time level $t=0$, the approximation above can be used to find solution values at the time level $t+k$. Having computed the solution at the $t+k$ time level, the solution at the $t+2k$ time level can be computed and continuing in this *time-stepping* method the solution at points throughout the domain can be obtained The formula can be written as follows:

$$u(x,t+k) \approx u(x,t) + \frac{k\,D}{h^2}\{u(x+h,t) - 2u(x,t) + u(x-h,t)\}$$

[1] Diffusion Equation
[2] Finite Difference Method

The simplest way of thinking about the finite difference method is to regard the *x-t* domain as a grid. With the initial information at *t=0*, we can set values to the nodes on the bottom row.



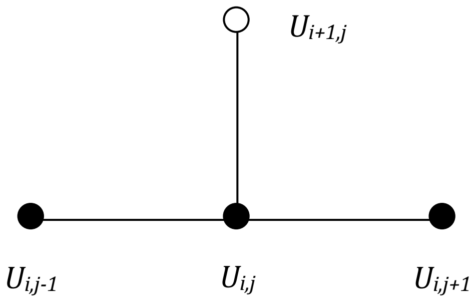
Given the values at the nodes on the bottom row, approximations to the values at the row above can be determined using the finite difference formula and this can be continued until we have approximations to the solution at all grid points.

If we let $U_{0,j} \approx u(jh,0)$ for $j=0..N$ represent the known initial values at the node points. We can then compute the values on the next and subsequent rows of nodes using the following formula, which is represents a discrete equivalent to the original diffusion equation:

$$U_{i+1,j} = U_{i,j} + \frac{k\,D}{h^2}\left(U_{i,j+1} - 2U_{i,j} + U_{i,j-1}\right).$$

Note that in order to arrive at the difference equation above, the approximation symbol has been replaced by an equals symbol. The results from the difference equation represents an equation that relates approximate values.

Finite difference methods are often illustrated using *molecules*. A molecule links the nodes for one application of the difference equation. For the finite difference method defined, the molecule has the following form:



Note that the nodes for the previously *known* values are shaded and the value computed within the molecule is shown as an empty node.

This is the simplest method for solving the diffusion equation. However, it is not the most accurate and it is stable only if ${kD}/{h^2} < \frac{1}{2}$ (*Courant–Friedrichs–Lewy* condition). There are many more methods, many of them with better convergence and stability properties.